Examen Analyse 2012-13

Datum : 08-02-2013
Tijd : 09.00 - 12.00, 5118.-156

You need to clearly provide arguments for all your answers; 'yes' or 'no' answers are not allowed.

Grading scheme: Total number of points: 100. Free: 10. Number of points is specified after each question.

1. (a) Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ uniformly continuous on A, that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in A$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. Suppose that $(x_n) \subseteq A$ is a Cauchy sequence. Prove that $f(x_n)$ is a Cauchy sequence. **(10 points)**

   (b) Is the statement of the previous question still true if the assumption of uniform continuity is replaced with continuity? Give a proof or counterexample to support your claim. **(5 points)**

2. (a) Let Λ be an indexing set (that is, its elements are subscripts), and suppose that for each $\lambda \in \Lambda$ the set $E_\lambda \subseteq \mathbb{R}$ is closed.
   Using just the definition of a closed set, prove that the intersection

$$E = \bigcap_{\lambda \in \Lambda} E_\lambda$$

   is closed as well. **(5 points)**

   (b) Let $E_1, E_2, \ldots, E_m$ be a finite collection of closed subsets of $\mathbb{R}$.
   Using just the definition of a closed set, prove that the finite union

$$E = \bigcup_{i=1}^{m} E_i$$

   is closed as well. **(5 points)**

   (c) Is the infinite union of closed sets necessarily closed? Give a proof or counterexample to support your claim. **(5 points)**

3. Let $A \subseteq \mathbb{R}$ and for $n \in \mathbb{N}$ let $f_n : A \to \mathbb{R}$. Suppose that each f_n is bounded on A and that (f_n) converges uniformly to some function $f : A \to \mathbb{R}$ on A.

   (a) Show that f is bounded on A. **(5 points)**

   (b) Show that (f_n) is uniformly bounded on A. That is, there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in A$ and $n \in \mathbb{N}$. **(10 points)**

4. For each of the following sequences of functions (f_n), find the pointwise limit on the given set A, and determine wether the convergence is uniform on A. Provide your claims with rigorous argumentation.

(a) $f_n(x) = x^{2n}, \quad A = [-1, 1]$. **(5 points)**

(b) $f_n(x) = \dfrac{\arctan(nx)}{n(x^2+1)}, \quad A = \mathbb{R}$. **(5 points)**

(c) $f_n(x) = \dfrac{1}{n} \log(x+1), \quad A = [0, \infty)$. **(5 points)**

5. Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n \sin(n\pi x)}{n^2}$$

(a) Show that f is continuous on $[0, 1]$. **(7 points)**

(b) Show that f is differentiable on $[0, 1)$. **(8 points)**
Hint: consider an arbitrary point $x_0 \in [0, 1)$.

6. Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that for any partition P of the interval $[0, 1]$ we have $L(f, P) = 0$. **(4 points)**

(b) For a given $\epsilon > 0$, construct a partition P_ϵ of $[0, 1]$ such that $U(f, P_\epsilon) < \epsilon$. **(8 points)**
Hint: how much discontinuities does f have on the interval $[\frac{\epsilon}{2}, 1]$?

(c) Show that f is integrable on $[0, 1]$ with $\int_0^1 f = 0$. **(3 points)**